

Analytical expression for wave scattering from exponential height correlated rough surfaces

M. Zamani¹, M. Salami², S. M. Fazeli³, G. R. Jafari¹ *

¹ Department of Physics, Shahid Beheshti University, G.C., Evin, Tehran 19839, Iran

² Department of Physics, Shahroud Branch, Islamic Azad University, Shahroud, Iran

³ Department of Basic Science, university of Qom, Qom, Iran

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Abstract

Wave scattering from rough surfaces in addition the inverse scattering is an interesting approach to obtain the surface topography properties in various fields. Analytical expression in wave scattering from some known rough surfaces, not only help us to understand the scattering phenomena, but also would prove adequate to be a criterion to measure the information for empirical rough surfaces. For a rough surface with an exponential height correlation function, we derive an analytical expression for the diffused part and expanded it in two asymptotic regimes. We consider one surface as slightly rough and the other as very rough based on the framework of the Kirchhoff theory. In the end, we have measured the role of various Hurst exponents and correlation lengths on scattering intensity in self-affine surfaces. We have shown that by increasing the Hurst exponent from $H = 0$ to $H = 1$, the diffuse scattering decreases with the scattering angle.

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1 Introduction

The study of rough surfaces is needed in many scientific fields. This is due to the fact that roughness is a scaling parameter and depending on the scale of observation which plays an important role in natural phenomena. The behavior of scattered fields from rough surfaces is of interest to many research topics, ranging from X-ray to radar wave in a long time in various fields. One of the most general approaches to study wave scattering is the Kirchhoff theory [1, 2, 3, 4, 5]. This theory is an electromagnetic theory and is known as a tangent plane theory which is most widely used to calculate the distribution of the specular and diffuse parts of the reflected light. The Kirchhoff theory treats any point on a scattering surface as a part of an infinite plane, parallel to the local surface tangent [3].

*Email: g_jafari@sbu.ac.ir

There are two types of problems in this area: (a) direct problem, (b) inverse problem. The direct problem is concerned with determining the scattered field from the knowledge of the incident field and the scattering obstacle [6, 7, 8, 9, 10, 11]. Interesting works in the context of wave scattering has been carried out Brewster's scattering angle [12] and the Rayleigh hypothesis [13] have been studied. In addition, Ingve et al [14] studied wave scattering from self-affine surfaces where Leskova et al [15] studied the coherence of p-polarized light scattered from rough surfaces. Since the wave-length could be either smaller or greater than the height fluctuations, the two scale theory was proposed [16]. In a further study, effects of interference of two beam scattering was considered [17]. Scattering from multilayer is studied by Carniglia [18]. Elson et al [19] investigated vector wave scattering theory to estimate the angular distribution of scattered light from optical surfaces. Some of the researcher studied shadowing effect on rough surfaces [20, 21]. The inverse problem is concerned with inverse scattering techniques to [22, 23, 24, 25] measure the statistical properties of rough surfaces. Sinha et al showed how scaling properties exists in X-rays and neutrons from rough surfaces [26], Jafari et al [27] obtained the surface roughness, Dashtar et al used inverse wave scattering to determine the height distribution on a rough surface [28].

The results indicate that when the wavelength is larger than the correlation length, the scattered wave differs with the case where the wavelength is smaller than the correlation length. In this work an analytical expression for the diffused intensity is obtained. For the case where the height correlation function has an exponential behavior. In addition scattering from self-affine surfaces is studied for various roughness exponents and correlation lengths.

The paper is organized as follows. In Sec. II we present a theoretical description of Kirchhoff theory and surface roughness is the source of scattered field. The self-affine fractal model of the surfaces in Sec. III. The comparison of coherent and diffuse parts of scattered intensity and scattering from self-affine surfaces with different parameters are described respectively in Sec. IV. Finally, some general conclusions are presented in Sec. V.

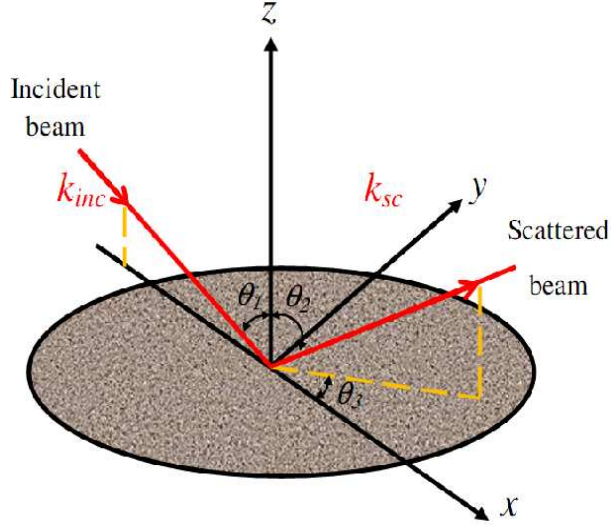


Figure 1: Schematic figure to show the geometry used for wave scattering from a rough surface.

2 Kirchhoff theory

In Kirchhoff theory, the incident field is written as $\psi^{inc}(r) = \exp(-ik_{inc}.r)$, where k and r are the wave number and position respectively. Fig. 1 shows schematically the scattering phenomena from a rough surface with dimensions $-X \leq x_0 \leq X$, $-Y \leq y_0 \leq Y$. The scattered field in r is denoted by ψ^{sc} and relies on three important assumptions: a) The surface is observed in the far field. b) No point on the surface has infinite gradient. Therefore, the Fresnel laws can be locally applied. c) The reflection coefficient, R_0 , is independent of the position on the rough surface. The total scattered field over the mean reference plane A_M , is given by [3],

$$\psi^{sc}(r) = \frac{ike^{ikr}}{4\pi r} \int_{S_M} (a \frac{\partial h}{\partial x_0} + b \frac{\partial h}{\partial y_0} - c) \times \exp(ik[Ax_0 + By_0 + Ch(x_0, y_0)]) dx_0 dy_0, \quad (1)$$

$$A = \sin \theta_1 - \sin \theta_2 \cos \theta_3,$$

$$B = -\sin \theta_2 \sin \theta_3,$$

$$C = -(\cos \theta_1 + \cos \theta_2),$$

$$a = \sin \theta_1 (1 - R_0) + \sin \theta_2 \cos \theta_3 (1 + R_0),$$

$$b = \sin \theta_2 \sin \theta_3 (1 + R_0),$$

$$c = \cos \theta_2(1 + R_0) - \cos \theta_1(1 - R_0).$$

The coherent and the diffuse intensities are given by [3],

$$\begin{aligned} I_{coh} &= I_0 e^{-g}, \\ \langle I_d \rangle &= \frac{k^2 F^2}{2\pi r^2} A_M e^{-g} \int_0^\infty J_0(kR\sqrt{A^2 + B^2}) \times [e^{gCor(R)} - 1] R dR, \end{aligned} \quad (2)$$

where I_0 is the scattered intensity from a flat interface. The parameter g is equal to $= k^2 \sigma^2 C^2$, where $C = \cos \theta_1 + \cos \theta_2$, and the height-height correlation function $Cor(R)$ which is definite for an isotropic surface ($\langle h(x) \rangle = 0$) is equal to $\frac{\langle h(x+R)h(x) \rangle}{\sigma^2}$.

3 Scattering from self-affine surfaces

3.1 Self-affine surface

The correlation function of self-affine surfaces is known by correlation length, ξ , which is the mean lateral length of surface features, e.g. grain size or other. Following Sinha et al. [26, 29, 30], can be approximated by an analytic function, $Cor(R) \approx e^{-(\frac{R}{\xi})^{2H}}$, which contains the following limits. For $R \gg \xi$, the correlation vanishes, ($Cor(R) = 0$) and for $R \ll \xi$, the correlation function would look like $Cor(R) \simeq 1 - (\frac{R}{\xi})^{2H}$, and roughness exponent H would be between zero and one. The large values of H correspond to smoother height-height fluctuations, while small values of H characterize irregularity in height for rough surfaces at the length scales ($R \ll \xi$) [26, 29, 30].

For a Gaussian correlation function ($H = 1$), Eq. (2) yielded an expression for the diffuse scattered intensity [3]:

$$\langle I_d \rangle = \frac{k^2 F^2 \xi^2 e^{-g}}{4\pi r^2} A_M \sum_{n=1}^{\infty} \frac{g^n}{n! n} \exp\left(-\frac{k^2(A^2 + B^2)\xi^2}{4n}\right). \quad (3)$$

3.2 Analytical expression for a rough surface with exponential height correlated

Knowing any exact results in wave scattering from rough surfaces is helpful to better understand the scattering phenomena. One of these surfaces is exponential height correlated surface which could be an

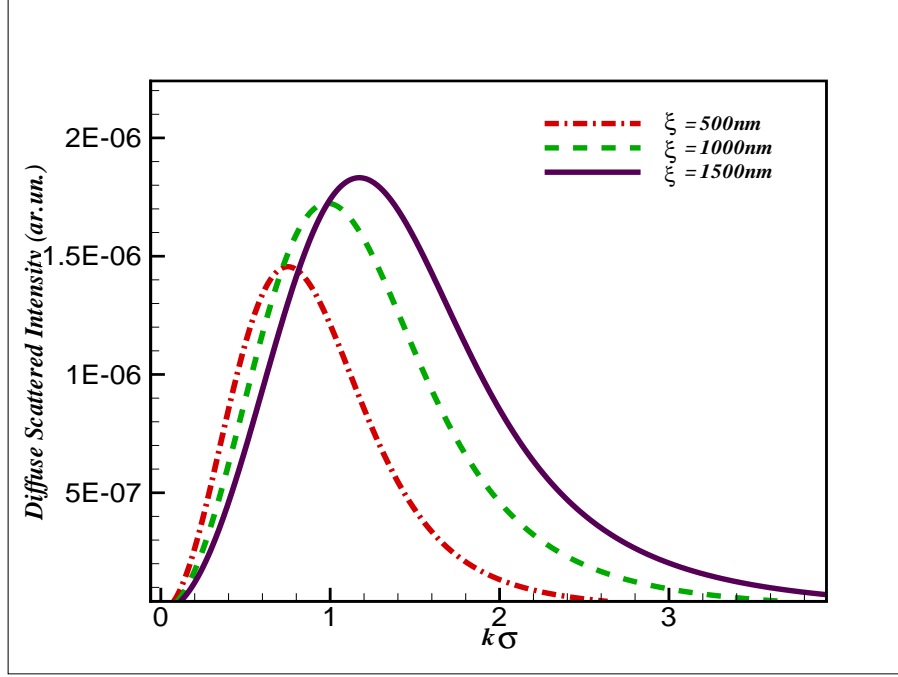


Figure 2: (Color online) Dependence of diffuse scattered intensity on $k\sigma$ for different correlation length ($\xi = 500, 1000, 1500nm$) and for angles $\theta_1 = \theta_3 = 0^\circ$, $\theta_2 = 20^\circ$, wavelength $\lambda = 500nm$ and $H = 0.5$.

important and suitable criteria to measure information content in other rough surfaces. We obtain an analytical expression for the correlation function $Cor(R) = e^{-(\frac{R}{\xi})^{2H}}$ with $H = 0.5$, which appears in an exponential form $Cor(R) = e^{-\frac{R}{\xi}}$. With regarding $e^{gCor(R)} = \sum_{n=0}^{\infty} \frac{g^n cor^n(R)}{n!}$, the diffuse part of the scattered intensity will be:

$$\begin{aligned}
\langle I_d \rangle &= \frac{k^2 F^2}{2\pi r^2} A_M e^{-g} \int_0^\infty J_0(kR\sqrt{A^2 + B^2}) \times \sum_{n=1}^{\infty} \frac{g^n cor^n(R)}{n!} R dR \\
&= \frac{k^2 F^2}{2\pi r^2} A_M e^{-g} \sum_{n=1}^{\infty} \frac{g^n}{n!} \int_0^\infty J_0(kR\sqrt{A^2 + B^2}) cor^n(R) R dR \\
&= \frac{k^2 F^2}{2\pi r^2} A_M e^{-g} \sum_{n=1}^{\infty} \frac{g^n}{n!} \int_0^\infty J_0(kR\sqrt{A^2 + B^2}) e^{-\frac{nR}{\xi}} R dR \\
&= \frac{k^2 F^2}{2\pi r^2} A_M e^{-g} \sum_{n=1}^{\infty} \frac{g^n}{n!} \frac{\xi^2}{n^2(1 + \frac{k^2(A^2+B^2)\xi^2}{n^2})^{\frac{3}{2}}}.
\end{aligned} \tag{4}$$

Eq. (4) enables the study of rough surfaces in two asymptotic limits:

A. Slightly rough surfaces, For slightly rough surfaces ($g \ll 1$), the series in Eq.(4) converges

quickly and only the first term needs to be considered. So, the diffuse field intensity becomes:

$$\langle I_d \rangle = \frac{k^2 F^2}{2\pi r^2} A_M e^{-g} g \frac{\xi^2}{(1 + k^2(A^2 + B^2)\xi^2)^{\frac{3}{2}}}. \quad (5)$$

B. Very rough surfaces, The total intensity is $\langle I \rangle = \langle I_{coh} \rangle + \langle I_d \rangle = e^{-g} I_0 + \langle I_d \rangle$, when the surface is very rough ($g \gg 1$), the coherent field will be negligible and the total intensity is equivalent to the diffuse ones [3] (Chap. 4, Page 92). In addition, we know that $I_0 \propto X \cdot \frac{\sin(kAX)}{kAX}$, when sample size is larger than wavelength, the coherent field appears just in specular angle. For the non-specular angle we have only diffuse field ($\lambda \ll X, Y$ or $kAX \gg 1 \Rightarrow X \cdot \frac{\sin(kAX)}{kAX} = \pi \cdot \delta(kA)$) [31]. So, according Eq.(2) the total scattered intensity which is approximately equal to one could be written as:

$$\langle I \rangle = \frac{k^2 F^2}{2\pi r^2} A_M \int_0^\infty J_0(kR\sqrt{A^2 + B^2}) e^{-g} (e^{gCor(R)} - 1) R dR. \quad (6)$$

When the length scale is larger than the correlation length ($R \gg \xi$), the correlation function vanishes ($Cor(R \gg \xi) = 0$). In this case, according to Eq. (6), there is no scattered intensity in the large length scale ($R \gg \xi$). For convergence, it is sufficient to extend the numerical integration from zero to 5ξ . In the category which $g \gg 1$ and for small range of R , Eq. (6) is equal to:

$$\begin{aligned} \langle I \rangle &= \frac{k^2 F^2}{2\pi r^2} A_M \int_0^\infty J_0(kR\sqrt{A^2 + B^2}) e^{-g[1 - Cor(R)]} R dR \\ &= \frac{k^2 F^2}{2\pi r^2} A_M \int_0^\infty J_0(kR\sqrt{A^2 + B^2}) e^{-g\frac{R}{\xi}} R dR \\ &= \frac{k^2 F^2}{2\pi r^2} A_M \frac{\xi^2}{(1 + \frac{k^2(A^2 + B^2)\xi^2}{g^2})^{\frac{3}{2}} g^2}. \end{aligned} \quad (7)$$

4 Effects of the correlation length and Hurst exponent on scattering intensity

4.1 Variation of Correlation Length

According to Eq. (2) the correlation function only effects the diffuse part leaving the coherent part unaffected. Fig. 2 compares the diffused scattering vs $k\sigma$, with a fixed $\lambda = 500$ nm, for three values of the correlation length $\xi = 500, 1000, 1500$ nm and the angles ($\theta_1 = \theta_3 = 0^\circ, \theta_2 = 20^\circ$). We have chosen the value of parameters in the way which are compatible with experimental setup. The numerical results

which were obtained from Eq. (2) are shown for $k\sigma = 0$ to 4. A characteristic feature seen in Fig. 2 is the presence of a maximum in the diffused part as a function of the $k\sigma$, which depends on correlation length. By increasing the correlation length the peak of the diffused part is shifted to large $k\sigma$. Indeed, small fluctuation have low effects on larger wavelengths. Note that decreasing wavelength and increasing rms have the same effect on wave scattering. Figuratively speaking, a surface with shorter correlation length appears rough to small wavelengths and smooth to large wavelengths. Therefore, in the context of scattering, $k\sigma$ is more suitable than k or σ alone. In other words the longer angles would play a more significant role on diffused intensity. This result in a decrease\increase of intensity for smaller\longer angles. In addition, the smaller correlation length changes its behavior as it shifts to larger $k\sigma$.

4.2 Variation of Hurst exponent

The Hurst exponent appears in the correlation function and the correlation function enters into the diffuse part. So, the variations of Hurst exponent could effect the diffuse scattered intensity, too. To more sense the role of Hurst exponent in wave scattering, we study diffuse scattering in two regimes. Fig. 3 depicts the variation of the diffuse scattering intensity with respect to scattering angle for different Hurst exponents for the two cases: $\lambda < \xi$ & $\lambda > \xi$. The first point in Fig. 3 is that by increasing H , the diffuse scattering intensity is reduced for angles close to the specular angle widening the curve. Thus it can be observed in larger angles. For non-specular scattering angle far from specular angle, this behavior is opposite, so by increasing H , the diffuse scattering intensity is increased. Another point is that for $\lambda < \xi$ the fluctuations of the surface are observed better and the roughness is increased so the diffuse scattering intensity increases (Fig. 3a). For the case $\lambda > \xi$, smaller fluctuations of the surface are not observed and surface seems smoother and the diffuse scattering intensity decreases sharply. This behavior can be seen by comparing Figs. 3a and 3b. Variation of the diffuse scattering intensity with respect to H has the same behavior for both wavelengths smaller and larger than the correlation length.

There are two kinds of behaviors in scattering intensity based on wavelength smaller or larger than the correlation length. When the wavelength is less than correlation length, the small height fluctuation

is being observed. So the wavelength acts as a scale for the observation of the surface. The more observed height fluctuation leads to the more diffuse scattered intensity. There is opposite behavior when the wavelength of incident wave is larger than the surface correlation length. In this case, the incident wavelength is not able to recognize small fluctuations on the surface and loses the scale of scattering angle. This means that the surface seems smooth for this scale of observation (wavelength). This results in a decrease in the diffuse intensity.

Fig. 4 shows the variation of the diffuse scattering intensity with respect to $k\sigma$ for different H for the non-specular scattering angles ($\theta_1 = \theta_3 = 0^\circ$ and $\theta_2 = 20^\circ$). By increasing H , the diffuse scattering intensity decreases, which is in good agreement with Fig. 3. In Fig. 4, by increasing H , the curve shifts to larger $k\sigma$.

In order to investigate the role of the Hurst exponent, a surface with the observed values of for the correlation length ($\xi = 1000nm$), and wavelength ($\lambda = 500nm$) is considered. When $k\sigma$ is small, either by means of small σ or large λ , the scattering is not sensitive to the corrugation, independently of the Hurst exponent. For the chosen correlation length of 1000 nm, the diffuse intensity in Fig. 4 is similar for $k\sigma < 0.4$, the influence of the Hurst exponent becomes apparent only for large $k\sigma$.

5 Conclusion

The roughness of a surface effects the scattered wave from rough surfaces. We have found an analytical expression for spectral scattered intensity for a rough surface with exponential height correlation function and investigated the results for asymptotic regimes of slightly rough and very rough surfaces. The exact solution helps us understand the scattering phenomena and could be a suitable criteria to measure the value of information in a unknown rough surface.

The role of the Hurst exponent has been investigated for self-affine rough surfaces. Such an examination was performed over a wide range of surface topographies, from logarithmic ($H = 0$) to a power-law self-affine rough surface, $0 < H < 1$. The roughness exponent H has a strong impact on the diffused part of wave scattering mainly for relatively large correlation lengths. Therefore, Hurst exponent must

be taken carefully into account before deducing the roughness correlation lengths from wave scattering measurements.

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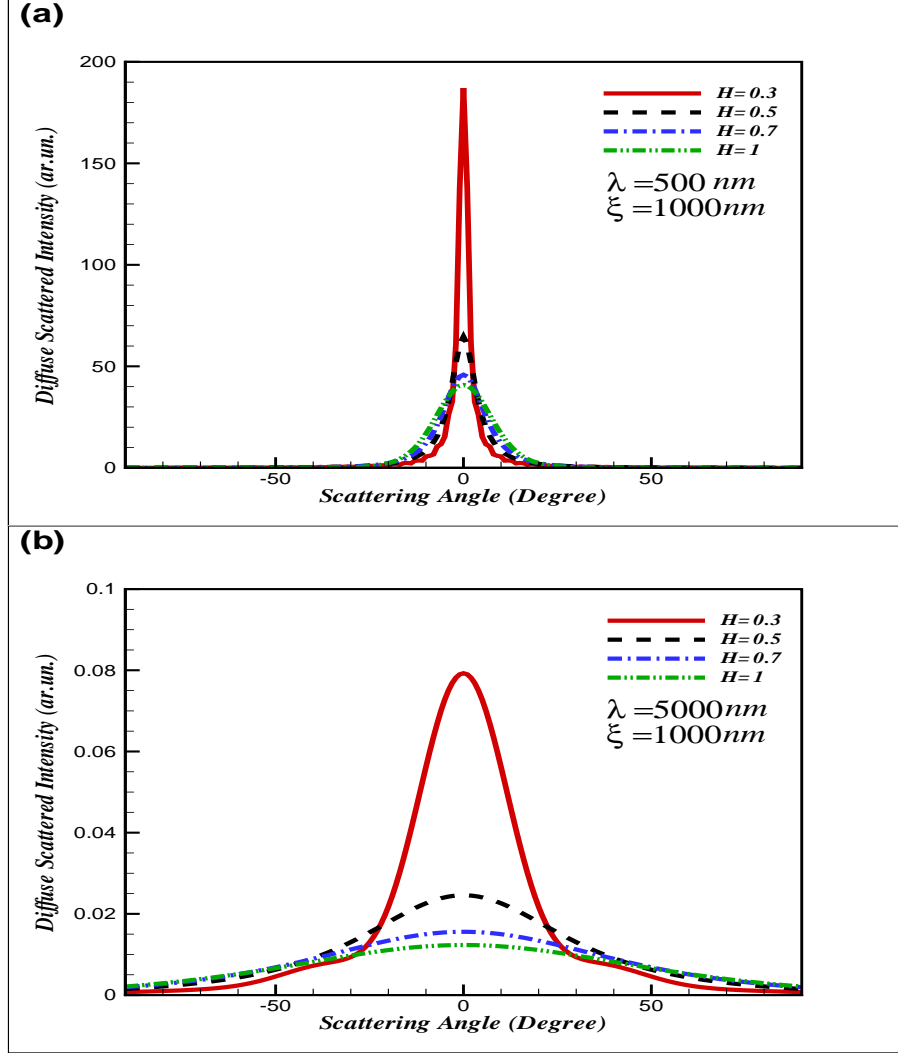


Figure 3: (Color online) Dependence of diffuse scattered intensity on θ_2 for angles $\theta_1 = 0^\circ$, $\theta_3 = 0^\circ$ and different Hurst exponent, correlation length $\xi = 1000 \text{ nm}$, standard deviation $\sigma = 50 \text{ nm}$ and the incident wavelength (a) $\lambda = 500 \text{ nm}$, (b) $\lambda = 5000 \text{ nm}$.

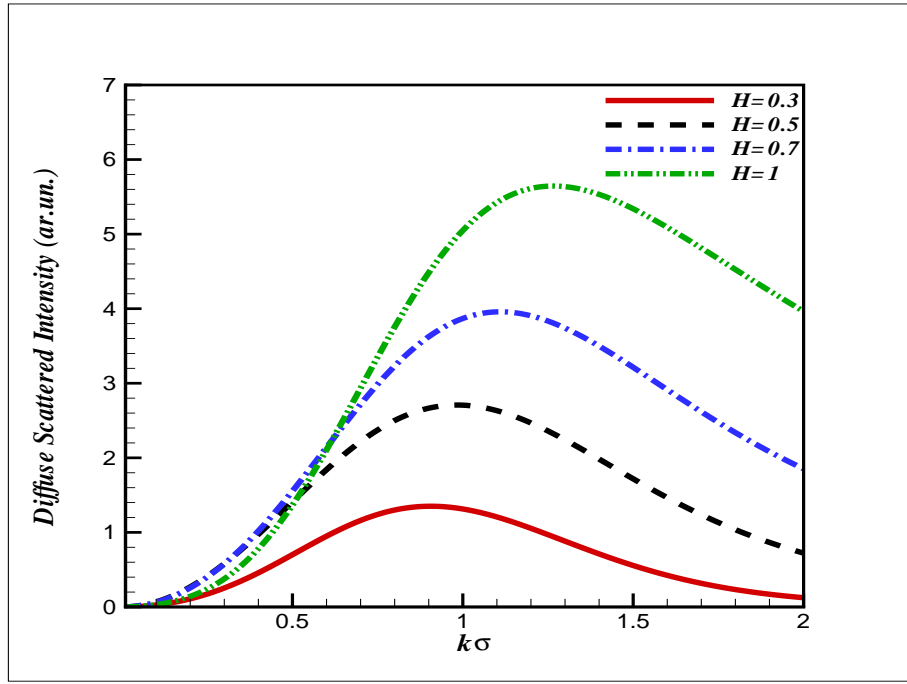


Figure 4: (Color online) Dependence of diffuse scattered intensity on $k\sigma$ for angles $\theta_1 = \theta_3 = 0^\circ$ and $\theta_2 = 20^\circ$ different Hurst exponent, correlation length $\xi = 1000nm$, wavelength $\lambda = 500nm$.